An Approximate Calculation of the Onset Velocity of Cavity Oscillations

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The velocity for the onset of oscillations in a cavity adjacent to a convecting field is calculated as a hydrodynamic stability problem. It is shown that the frequency at the onset of the oscillations corresponds closely to one of the natural frequencies of the energy storage system of the cavity. The results are in general agreement with experimental data for acoustic and hydroelastic oscillations.

Nomenclature

a = speed of sound
F = arbitary potential
$f_n(s)$ = coupling function [see Eq. (25)]
G = Green's function
H_o = Hankle function of zeroth order and first kind
i = $(-1)^{1/2}$
J_o = Bessel function of zeroth order and first kind
k = wave number
k_m = modified wave number defined below [Eq. (3)]
$l_{x,y,z}$ = cavity lengths in x, y, $(-)z$, direction
M = Mach number
N_0 = Bessel function of zeroth order and second kind
p = static pressure
q = q_{cx}/U_∞
q_{cx} = x component of cavity velocity at interface ($z = 0$)
$\mathbf{r}^s$ = position vector of interface surface
S = Strouhal number = $\omega l_x/U_\infty$
t = time
δu = x component of perturbation velocity
U_∞ = main flow velocity outside the cavity, in $+x$ direction
x,y,z = coordinates z is + into the main flow, x is downstream, and y completes a right-handed system
δ = maximum displacement of ζ normalized by wavelength
ρ = static density
Φ = potential for outer flow
ϕ = perturbation potential for outer flow normalized with respect to U_∞
ζ,σ = shape of interface
Θ = cavity natural frequency function [Eq. (24)]
ψ = perturbation potential in cavity normalized with respect to U_∞
ω = angular frequency rate

Subscripts

ϕ = outer flow property
c = cavity flow property
∞ = static conditions far from the cavity
o = source variable in Green's function
n = unperturbed natural frequency of cavity, nth mode

Introduction

A CAVITY or cut-out in a moving object can generate sound, although the detailed circumstances for which a given cavity will or will not act this way remains to be fully described. The physical phenomena behind this oscillation was studied experimentally as early as 1878 by Strouhal (who called the phenomena "reibungstone") and by Kohlrausch in 1881. Rayleigh[1] in discussing the causes for the oscillations says that "... their production also is doubtless connected with the instability of vortex sheets ..." He concludes the discussion with the remark "... It would also be of interest to extend the experiments to liquids."

Rayleigh also presents results obtained by the use of hydrodynamic stability theory which show (following Helmholtz) that infinitely long vortex sheets are unstable, i.e., any disturbance causes the motion of the vortex sheet to increase rapidly. In the case of particular interest here, the flow induced oscillation by a hole, or cut-out has generally been attributed to the instability of vorticity in one form or another[2-6]; in Ref. 6 Harrington and Dunham state that such an instability can be found in liquids (water) as well as in gases (air). This supports the validity of Rayleigh's suggestion that at low speeds these phenomena are essentially vortex driven.

Here the cavity behavior is studied by a simultaneous solution of the flow process for the external and internal flows. The geometry of the situation is shown in Fig. 1. The common area of the outside flow and the inside flow is the x-y plane in the region $0 \leq x \leq l_x$, $0 \leq y \leq l_y$. The experimental data discussed below indicates that this process is essentially two dimensional. Thus the limit in the y direction will be allowed to increase without limit in both the positive and negative direction. In this way the problem is made two dimensional.

In the spirit of classical fluid mechanics, the fluid will be assumed to be inviscid, and perfect. The motion of the interface must always coincide with the motion of the fluid particles that happen to be at the interface. The normal velocity and the pressure are continuous across the interface but the tangential velocity may be discontinuous across the cavity interface. Hence the vortex sheet previously mentioned seems to be a reasonable model. The leading edge of the cavity, the y axis, is also the trailing edge of the solid wall ahead of the cavity. Hence a Kutta-condition will be

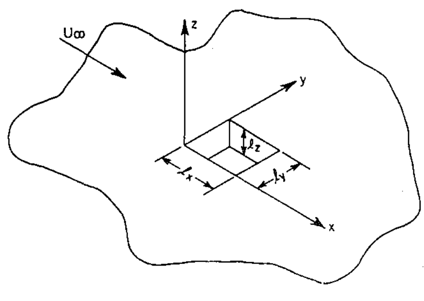

Fig. 1 Configuration to be studied.

Received October 2, 1969; revision received June 9, 1970. This research was carried out under the Bureau of Ships, Fundamental Hydromechanics Research Program, S-R-009 01 01, administered by the David Taylor Model Basin. The author would like to thank M. Landahl of the Department of Aeronautics and Astronautics, Massachusetts Institute of Technology for his interest and criticism.

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imposed on the y axis; indeed, as it should be imposed at each sharp trailing edge.

The influence of walls on the vorticity can be shown in terms of classical vortex kinematics.[7] An isolated vortex in the presence of a solid wall tends to move parallel to the wall with a velocity less than that induced by the vortex and its image but in the same direction. Two limiting two-dimensional cases appear at once.

1) A deep cavity is one whose dimension normal to the external stream greatly exceeds the dimension in the stream direction. The vorticity is primarily governed by the images in the forward and aft walls. Therefore, there is a tendency for the vortex motion to be in the direction of the depth.

2) A shallow cavity is one whose dimension in the stream direction is much greater than the dimension in the normal direction. Here the vorticity is primarily governed by the images along the bottom (the long dimension again) so the vortex motion tends to be in a fore and aft direction.

It can be inferred that in three dimensions the lateral motion would tend to be suppressed or converted into depth motions for deep cavities, whereas it would tend to remain lateral in the shallow cavities. These inferences are supported by observation of both shallow cavities[16] and deep cavities.[17,18]

In the steady state (no oscillation) the vortex sheet is convected at the mean velocity between the outer flow (U_∞) and the inner flow (q_{cx}), and the velocity induced by the image of the vorticities in the walls. The vertical wall at the leading edge of the cavity tends to distort the vortex sheet downward, whereas the wall at the trailing edge tends to bend the vortex sheet upward.

If this vorticity is perturbed to move at some other speed, the vorticity will exert a force at right angles to the plane defined by the vortex elements and the velocity vector. This force results in an induced velocity in the fluid. The product of the force and the induced velocity is the rate work is done on the cavity by the perturbed vorticity. If the rate of doing work is less than the ability of the cavity to dissipate energy by viscosity or radiation of energy from the cavity, perturbations will be damped. At the critical velocity energy supplied just equals the energy radiated. Above the critical velocity, the flow does work on the cavity, increasing the intensity and the frequency of the oscillation until the energy dissipated equals the energy supplied. Two other valid inferences may be drawn from the simplified model.

1) In order for the frequency to be less than the lowest natural frequency of the cavity energy storage system, some periodic excitation must exist. At high Reynolds numbers this is not likely to occur. Thus the lowest value of ω at which the oscillation will start is of the order of the lowest natural frequency of the cavity energy storage system.

2) It should be noted that this hypothesis indicates that the boundary condition on the pressure is more nearly $p_\phi + \rho_\phi(U_\infty - q_{cx})\delta u = p_c$ than $p_\phi = p_c$. However, since the boundary condition involves absolute pressure levels, the additional pressure is negligible when compared with the other pressure levels, even at 175 db (re 0.0002).

The approach to be followed here consists of expressing the perturbation potential for the outer flow in terms of an integral over the cavity-main-flow interface by means of Green's theorem. The flow in the cavity can also be expressed in this way. These two formal representations for the respective flow are both valid on the interface. Manipulation of these two formal solutions to satisfy the boundary conditions on the interface leads to an integral equation.† The characteristic equation of this integral equation provides a relation between the wave number and the frequency. For a real wave number the relation can be satisfied only for certain values of frequency. These values of frequency may be real or complex. Thus if the allowed frequencies are real, corresponding to real wave numbers, the perturbation is neutrally stable, i.e., it neither grows or decays. However, if corresponding to a real wave number, the frequency is complex, the stability is governed by the sign of the imaginary part of the frequency. If the sign of the imaginary part of the frequency is positive the perturbation grows and conversely, a negative sign indicates a decaying perturbation. The process is said to become unstable at those values of the parameter where the positive imaginary part of the frequency makes its first appearance. The statement that the process is unstable should not be taken to mean that the perturbation will grow without bound; only that the process departs from the initial state. It is possible that the unstable perturbation will lead to a new state that is stable. This possibility can only be investigated by means of the full equations. For the purpose of this calculation the desired results of the analysis is to provide a description (graphically or algebraically) of the onset of instability i.e., a zero imaginary part of the frequency in terms of the other parameters of the problem.

The condition so found can be expressed in terms of a Strouhal number, ($S = \omega l_x/U_\infty$) that is a function of Mach number, the cavity geometry and the flow in the cavity. The real part of the frequency corresponds closely to the frequency of the particular cavity energy storage mode, the value of S can be used to calculate the value of U_∞ at which the first instability takes place. Results are presented for a deep acoustic-room oscillator and hydroelastic oscillator.

Analysis

I. Main Flow Region

In the discussion that follows the velocity will be defined as the negative gradient of the velocity potential. The potential of the outer flow Φ can be represented in the form

$$\Phi = -U_\infty[x + \phi(x,y,z,t)] \quad (1)$$

The perturbation is assumed to be irrotational to at least second order. The basic equation for the potential is derived in several sources,[8–10] but the best form for this analysis is given by Garrick in Ref. 10. In terms of an arbitrary potential F and the local speed of sound a_F, the primary equation can be written in the form

$$\nabla^2 F = 1/a_F{}^2[\partial^2 F/\partial t^2 + \partial/\partial t(\nabla F)^2 + (\nabla F)\cdot(\tfrac{1}{2})(\nabla F)^2] \quad (2)$$

This equation can be linearized following Garrick's arguments and in terms of the perturbation ϕ, the linearized equation‡ is

$$\nabla^2\phi - 1/a_\infty{}^2[\phi_{tt} + 2U_\infty\phi_{xt} + U_\infty{}^2\phi_{xx}] = 0 \quad (3)$$

The boundary conditions on ϕ are 1) $z > 0$ outgoing radiation,[11] i.e., for a suitably modified wave number $k_m = k[1 - M\cos(n,x)]$

$$\lim_{r\to\infty} (r)^{1/2}[\partial\phi/\partial r - ik_m\phi] \to 0 \quad (4)$$

where

$$r = (x^2 + z^2)^{1/2}$$

2) On $z = 0$ suppose that the shape of the interface is

$$\zeta = \sigma(x,t) \quad (5)$$

† The solution of the basic equations involves the product of two functions. One of these functions contains only spatial variables and the other is the exponential function of frequency times time.

‡ It is sufficient for linearlization of the differential equation if $|M_\infty{}^2 - 1| \gg 0(\delta^{2/3})$ or $(\omega l_x/U_\infty) \gg 0(\delta^{2/3})$ where δ is the surface displacement, normalized with respect to the wavelength (see Ref. 10, p. 661).

The interface boundary condition is, on the outside,

$$\partial\zeta/\partial t = \partial\sigma/\partial t + U_\infty \partial\sigma/\partial x \tag{6}$$

so the vertical velocity outside the interface is

$$-\partial\phi/\partial z = (1/U_\infty)(\partial\sigma/\partial t) + \partial\sigma/\partial x \tag{7}$$

Thus the boundary conditions become

$$\begin{aligned} -\partial\phi/\partial z &= (1/U_\infty)(\partial\sigma/\partial t) + \partial\sigma/\partial x \quad 0 \leq x \leq l_x \\ \partial\phi/\partial z &= 0 \quad x \geq l_x, x < 0 \end{aligned} \tag{8}$$

Since a periodic solution is being sought the time factor can be suppressed by writing

$$\phi(x,y,z,t) = \phi'(x,y,z)e^{-i\omega t} \tag{9}$$

where ϕ' is complex, and where it is understood that the notation $\phi' e^{-i\omega t}$ is interpreted as the real part of the complex function $\phi' e^{-i\omega t}$. In this way $\partial^n/\partial t^n = (-i\omega)^n$. The prime in ϕ' is dropped for convenience.

If G_ϕ is the solution of the adjoint of the non-self-adjoint Eq. (3)[11,12] with a unit source, the potential for the outer flow can be written formally as (Ref. 12, p. 806)

$$\phi = -\frac{1}{4\pi}\int_0^{lx} G_\phi(\mathbf{r},\mathbf{r}_o{}^s)\frac{\partial\phi}{\partial z_o}dx_o \tag{10}$$

II. Cavity Flow Region

Inside the cavity a rotational flow pattern exists that is a result of the viscous interaction between outer and inner fluids. This pattern will be represented by a velocity $\mathbf{q}_c$ which is assumed to be unchanged by the oscillating flow. The perturbation to the rotational flow is assumed to be represented by the potential

$$\Psi = -U_\infty\psi(x,y,z,t) \tag{11}$$

The linearized equation for ψ governing the flowfield in the cavity with velocity $\mathbf{q}_c$ in terms of the speed of sound in the cavity a_c is

$$\nabla^2\psi - 1/a_c{}^2[\partial/\partial t + q_c\cdot\nabla^2]\psi = 0 \tag{12}$$

where the term q_c is assumed to satisfy a basic flow equation and

$$1/a_c{}^2\mathbf{q}_c\cdot[(\nabla\psi)\cdot\nabla\mathbf{q}_c] \ll 1 \tag{13}$$

This condition is compatible with the assumption that the basic internal convection process is not influenced by the perturbation. Note that $\mathbf{q}_c$ is treated as constant with respect to the operations ∇ and $\partial/\partial t$.[10] The sound speeds outside the cavity a_∞ and inside the cavity a_c are constant as a consequence of the linearization.

The boundary conditions for ψ are§

$$\partial\psi/\partial z = 0 \quad \text{on } z = -l_z \quad 0 \leq x \leq l_x \tag{14}$$

for acoustic cavity, and $\partial\psi/\partial z$ is given for hydroelastic cavity

$$\partial\psi/\partial x = 0 \quad \text{on } x = 0 \text{ and on } x = l_x \quad -l_z \leq z \leq 0 \tag{15}$$

for the acoustic cavity and $\partial\psi/\partial x$ is given for the hydroelastic cavity

$$\begin{aligned} \partial\psi/\partial z &= (1/U_\infty)(\partial\sigma/\partial t) + (q_{cx}/U_\infty)(\partial\sigma/\partial x) \\ &\text{on } z = 0 \text{ and } 0 \leq x \leq l_x \end{aligned} \tag{16}$$

§ The cavity possesses two possible mechanisms for storing energy. Energy can be stored in the walls by elastic deformation or energy can be stored by the fluid in the cavity if the fluid is compressible. In the actual case both effects are present. If the natural frequencies are well separated these two effects can be treated separately. Thus for this approximation the acoustic case is that with a compressible fluid and rigid walls; the hydroelastic case is flexible walls and incompressible fluid.

In terms of the interior Green's function G_c the formal solution is, after suppressing the time factor

$$\psi = \frac{1}{4\pi}\int_0^{l_x} G_c(\mathbf{r}_c,\mathbf{r}_o{}^s)\frac{\partial\psi}{\partial z_o}dx_o \tag{17}$$

to the approximation that the $\partial/\partial n \sim \partial/\partial z$.

All other surface integrals vanish because of the boundary conditions. Here the notation $(\ )_c$ means the solution in the cavity, $(\ )_\phi$ the solution outside and $\mathbf{r}_o{}^s$ is the position vector on the interface surface ($z = 0$).

III. Combined Solution

These formal solutions are converted to an integral equation by the following steps: 1) The vertical velocity occurring in each integrand is known in terms of the surface equation [Eq. (7) and (16)]. 2) The outer and inner pressures (Eq. 18) and (19) below, are computed from the formal solutions [Eq. (10) and (17)]. 3) The position coordinates in the pressure equation are allowed to approach the interface ($z = 0$). 4) These pressures are equated leading to an integral equation for the interface.

The pressure must be continuous across the opening. As $z \to 0^+$, $0 \leq x \leq l_x$ the static pressure can be found from the linearized Bernoulli equation,

$$p_\phi = p_\infty + \frac{1}{2}\rho U_\infty{}^2\left[-\frac{2}{U_\infty}\left(\frac{\partial\phi}{\partial x} + \frac{1}{U_\infty}\frac{\partial\phi}{\partial t}\right)\right] \tag{18}$$

The pressure on the inside, $z \to 0^-$

$$p_c = H(t) + \frac{1}{2}\rho U_\infty{}^2\left(\frac{2}{U_\infty}\right)\left[q_{cx}\frac{\partial\psi}{\partial x} + \frac{1}{U_\infty}\frac{\partial\psi}{\partial t}\right] \tag{19}$$

The $H(t)$ corresponds to the static pressure on the inside. If the process starts from a nonzero value of U_∞, $H(t)$ may be taken as zero. After carrying out all these steps and if the terms in $\partial\sigma/\partial x_o$ are integrated by parts one obtains

$$\begin{aligned} &\int_0^l \left[\frac{\partial}{\partial x_o}\left\{G_\phi - q_oG_c + \frac{iU_\infty}{\omega}\left(\frac{\partial G_\phi}{\partial x} - iq_o{}^2\frac{\partial G_c}{\partial x}\right)\right\} + \right. \\ &\left.\frac{\omega}{iU_\infty}(G_\phi - G_c) + \frac{\partial G_\phi}{\partial x} - q_o\frac{\partial G_c}{\partial x}\right]\sigma(x_o)dx_o - \\ &\left\{G_\phi(x,0,x_o,0) - q_oG_c(x,0,x_o,0) + \frac{iU_\infty}{\omega}\left(\frac{\partial G_\phi}{\partial x} - \right.\right. \\ &\left.\left. iq_o{}^2\frac{\partial G_c}{\partial x}\right)\right\}\sigma(x_0)\Big|_{x_o=0}^{l} = 0 \end{aligned} \tag{20}$$

The integrated term vanishes under the assumption that $\sigma(0) = \sigma(l_x) = 0$. Note that since

$$q_o = q_{cs}{}^{(x)}/U_\infty \tag{21}$$

q_o is just q_c evaluated on $z = 0$. The solution of Eq. (20) will provide a description of $\sigma(x,\omega)$ for $0 \leq x \leq l$. This solution will only exist for a certain relation between ω and the remaining parameters, say $\omega = \omega(l_x,U_\infty,M_\infty,q,\omega_n)$. The implications of real of complex values of ω have been discussed previously.

The Green's function for the external flow evaluated on the surface $z \to 0$ may be written¶

$$G_\phi = 2\pi i \exp{}^{i\frac{\omega M_\infty}{a_\infty}\frac{x-x_o}{1-M_\infty{}^2}} H_o\left(\frac{\omega|x - x_y|}{a_\infty(1 - M_\infty{}^2)}\right) \tag{22}$$

¶ The solution for G_ϕ and G_c may be deduced using Refs. 9, 12 or from pp. 554 et seq., 728 et seq. of Morse, P. M. and Ingard, K. U., *Theoretical Acoustics*, McGraw Hill, New York, 1968, and Ref. 14. The computation is not difficult but is lengthy. Interested readers may request from the author a copy of M.I.T. Aerophysics Report 38, as long as they are available, if they wish to see the details of the computations for G_ϕ, G_c and the infinite determinant.

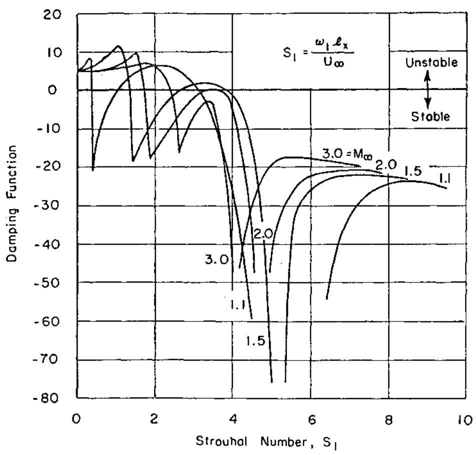

Fig. 2a Flow damping function, $M > 1$.

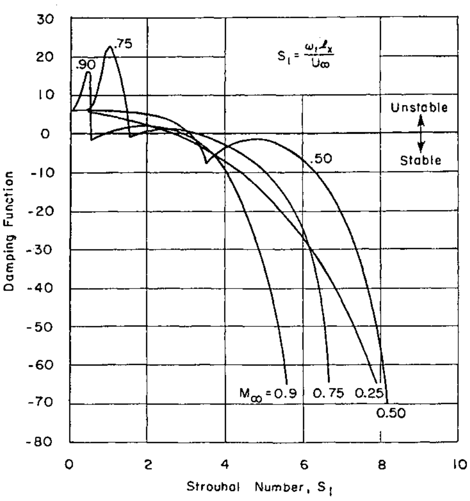

Fig. 2b Flow damping function, $M < 1$.

H_o is a Hankel function of the first kind. The Green's function in the cavity is made up of parts corresponding to energy storage in the elastic walls through bending and tension, and acoustic energy storage by the compressibility of the gas. The coupling between the walls and the fluid is best handled using the method of Ref. 14, which leads to an eigenfunction expansion. For the acoustic oscillations, the energy storage depends upon the speed of sound in the cavity, which is generally different from the freestream sound speed.

The nontrivial solution of the integral equation is found by use of a Fourier Series,[13] namely

$$\sigma(x) = \sum_{n=1}^{\infty} A_n(\omega) \sin\frac{n\pi x}{l_x} \qquad (23)$$

The next step is to expand the Green's functions in a double sine series. This process leads to an infinite number of equations in an infinite number of unknowns A_n. A solution can be found if the determinant is set to zero. It can be shown that the main diagonal dominates the roots of this determinant. This implies that the frequencies without convection present are a good first approximation to the solution [see footnote preceding Eq. (22)]. Thus the excitation frequency is approximately the undamped natural frequency of the lowest frequency of the energy storage system plus a correction term. Since the approximation of the determinant is incomplete, the approximate solution for the onset velocity should be functionally correct, but it may be in error by a numerical factor. In terms of the Strouhal number ($S = \omega l x/U_\infty$) and the cavity effect Θ_n the complex frequency is written

$$S = \omega_n l_x/U_\infty + \Theta_n \bar{q}_o{}^2\pi^2/2f_n(S) \qquad (24)$$

where $\bar{q}_o$ is the mean value of q. ω_n is the nth natural frequency of the cavity alone evaluated at cavity recovery temperature. The waves will be assumed to be acoustic even if strong.[15] $\Theta \doteq (l_x/l_z) \ll 1$, acoustic cavities for all ω_n. $\Theta \doteq (S/S_1)$ cavity with a membrane bottom.

$$f_n(S) = -\left[\pi - 4\lambda l_x \sin\kappa l_x + \frac{4(\kappa l_x + n\pi)\cos\kappa l_x}{(\lambda l_x)^2 - (\kappa l_x + n\pi)^2} + \frac{4(\kappa l_x - \pi)\cos\kappa l_x}{(\lambda l_x)^2 - (\kappa l_x - n\pi)^2}\right] \cdot \left[\frac{1}{[(\lambda l_x)^2 - (\kappa l_x + n\pi)^2]^{1/2}} + \frac{1}{[(\lambda l_x)^2 - (\kappa l_x - n\pi)^2]^{1/2}}\right] \cdot [S^2 - (n\pi)^2] + S\,\frac{(-1)^n 8n\pi \cos\kappa l_x}{[(\lambda l_x) - (\kappa l_x)]^2 - (n\pi)^2}\,J_o(\lambda l_x) \qquad (25)$$

where $$\lambda l_x = \frac{SM_\infty}{1 - M_\infty{}^2};\quad \kappa l_x = M_\infty \lambda l_x$$

If either term containing a radical becomes imaginary that term should be multiplied by the expression in square brackets

$$\left[\frac{4}{i}\left(\frac{\pi}{2} - \sin\kappa l_x\right) + \frac{4i(\kappa l_x + n\pi)\sin\kappa l_x}{(\lambda l_x)^2 - (\kappa l_x + n\pi)^2} - \frac{4i(\kappa l_x - n\pi)\sin\kappa l_x}{(\lambda l_x)^2 - (\kappa l_x - n\pi)^2}\right]$$

given in the preceding instead of that in square brackets in Eq. (25). The resulting i from the radical cancels the i in these terms to give a real part. Asymptotic forms for the perturbation are given in Appendix A.

S can be set equal to S_1 in $f(S)$ without further loss of accuracy. The imaginary part of $f(S_1)$ is plotted in Figs. 2a and 2b for subsonic and supersonic external flowfields, respectively.** In the region where the function is positive, perturbations may be expected to increase in amplitude. If a particular energy storage process corresponding to ω_n and a particular cavity length, l_x are selected, then Figs. 2a and 2b provide information about the stability as a function of U_∞ and M_∞. For incompressible flow (the same line as $M_\infty = 0.25$) the flow is unstable if the velocity exceeds a critical velocity determined at the zero crossing in Fig. 2a. As the external Mach number is increased, the behavior changes to include two unstable regions separated by a stable region. Further increases in Mach number result in a function which crosses the axis only once.

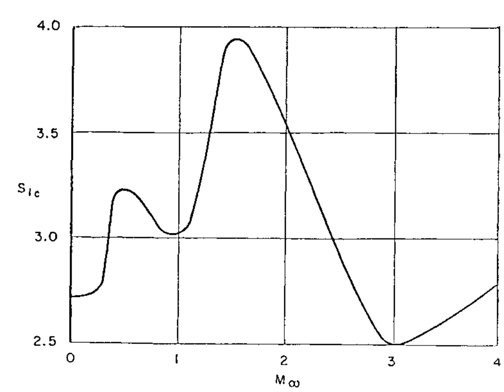

Fig. 3 Strouhal number for neutral stability vs Mach number.

** The calculations were performed at the Computation Center, Massachusetts Institute of Technology.

In the intermediate range of Mach numbers it is quite possible for several separate modes to be excited at the same time as shown in the applicable data in Refs. 16–18.

Figure 3 shows a plot of the Strouhal number of the lowest frequency for neutral stability, denoted S_{1c}, as a function of Mach number. For a M_∞ of about 3.0, the value of S_{1c} is approximately π if ω_1 is expressed in radians per second, or 0.50 if ω_1 is expressed in cycles per second. The critical Strouhal number S_{1c} has no asymptote for large M_∞ but rather S_{1c} continues to increase as $\ln M_\infty$.

This function S_{1c} has only one unknown, the freestream velocity U_∞. Hence Fig. 3 defines the velocity at which the process is neutrally stable.

Examples

I. Simple Open Ended Cavity

As a first check on these formulas consider the case of an open cavity in still air, so that $U_\infty = 0$ (but $q = 1$). In the case $l_x \ll l_z$ the natural frequency becomes

$$\omega \simeq \omega_1\left(1 - \frac{\pi^2}{8}\frac{l_x}{l_z}\right) - i\,\frac{\pi^3}{16}\frac{l_x}{l_z} \qquad \pi^3\frac{l_x}{l_z} \ll 1 \tag{26}$$

The effect of radiation can be considered as an increase in the depth of the tube, the numerical amount being $\Delta l_z \simeq \pi^2/8l_x$ (if $n = 1$). Rayleigh finds for a round tube a correction of 0.82R (Ref. 1, Appendix A—addition to Sect. 307). Converting to a hydraulic radius ($R' = 2A/P$) and allowing $l_y \rightarrow \infty$, the above result gives $R' = l_x$. Thus the effective length change calculated from Eq. (26) is in agreement with Rayleigh, except that the numerical value found by the elementary calculation is about 1.4 times too large. If l_x/l_z is small, this error is not serious.

II. Acoustic Cavity

The behavior of the acoustic cavity can be studied using a more restricted formula in the limit for small Mach numbers

$$S \simeq S_{10} \times \left\{1 + \frac{1}{2}\,\frac{\pi(l_x/l_z)\bar{q}_o^2}{(S_{1c}/\pi^2)J_o(\omega_1 l_x/a_\infty) - (16/\pi^2)S_{1c}N_o(\omega_1 l_x/a_\infty)}\right\} \tag{27}$$

where

$$S_{10} \sim \pi + 2\,\frac{(1 - M_\infty^2)^{1/2}}{\pi^2}\,J_o\left(\frac{\omega_1 l_x}{a_\infty}\right) \tag{28}$$

This result suggests that the performance of the acoustic cavity can be estimated from the product of a function of M_∞ and a function of (l_x/l_z). This suggestion can be tested against experimental data, although the Mach number range is wider than that for which Eqs. (27) and (28) are valid approximations. A series of cavities with different values of l_x/l_z were tested for different external Mach numbers.[17] The pertinent data from Ref. 17 for cavities $1/2 < l_x/l_z < 1$, which had a variety of widths, were reduced as follows. The observed Strouhal number was plotted against Mach number and was normalized with respect to a Mach number of three. The result was plotted against Mach number (Fig. 4).†† Also the theoretical curve from Fig. 3 is plotted on Fig. 4 after normalization with respect to $M_\infty = 3$. The measured Strouhal numbers were divided by the Mach number function of Fig. 4 and the results plotted against l_x/l_z. The result, shown in Fig. 5, suggests a simple linear dependence upon l_x/l_z.

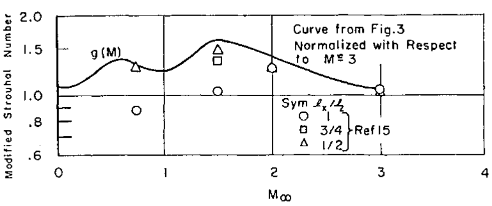

Fig. 4 Experimental Mach number function.

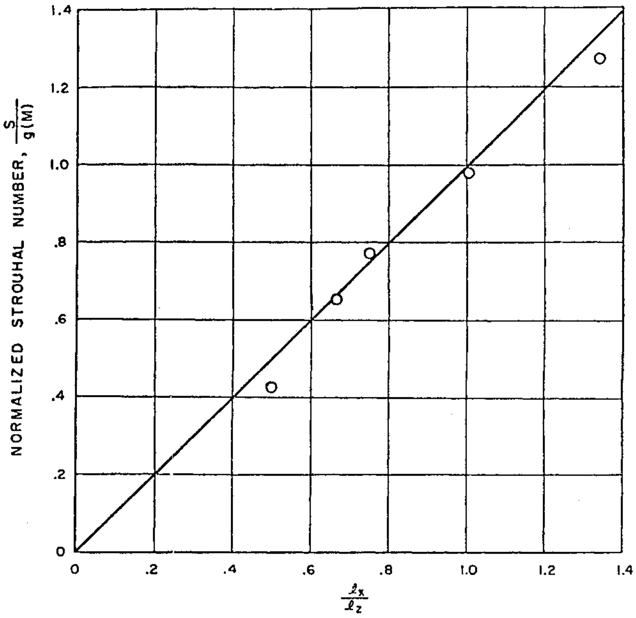

Fig. 5 Dependence of oscillation frequency of acoustic cavities upon l_x/l_z.

III. Hydroelastic Cavity

The case of the cavity with the membrane at its bottom can be studied by setting

$$\Theta_1 = S/S_1 \tag{29}$$

Thus in low speed flow

$$S \simeq S_{10}\left\{1 + \frac{1}{2}\,\frac{\pi^2\bar{q}_o^2}{S_{10}\,\dfrac{16}{\pi}\left[J_o\left(\dfrac{\omega_1 l_x}{a_\infty}\right) - N_o\left(\dfrac{\omega_1 l_x}{a_\infty}\right)\right]}\right\} \tag{30}$$

If the fluid is incompressible ($a_\infty \rightarrow \infty$) the term in the curly braces vanishes [$N_o(0) \rightarrow \infty$] so that

$$S \rightarrow S_{10} \simeq 0.43 \tag{31}$$

when the Strouhal number is expressed in terms of cycles per second. The data given in Ref. 10 indicates that at the onset of the instability the Strouhal number is about 0.30.‡‡ Further, Eq. (31) indicates that the critical Strouhal number is constant. This result agrees with the hydroelastic data given in Refs. 5, 6, and 19.

†† The normalization is necessary because the value of $\bar{q}_o^2$ is unknown. Since $\bar{q}_o^2$ is induced by the outer flow, the size of $\bar{q}_o$ is probably dependent upon Reynolds number.

‡‡ It will be remarked that the value of 0.43/0.30 is 1.43. This is about the same numerical factor encountered in the other cases. These two results suggest that use of the first approximation to the characteristic equation fails to give the correct numerical coefficient, but provides a description of the physical processes that is reasonable.

Closing Remarks

The results of the analysis show that at a critical velocity the frequency of excitation corresponds closely to the natural frequency without external convection. Below this velocity there is no oscillatory motion. This result agrees with the experimental evidence. The analysis also shows that in a hydroelastic system the Strouhal number at the onset of instability is essentially a constant, as has also been noted in the experiments. Hence one way to delay the onset of instability is stiffening the energy storage system to increase ω_n.

This calculation fails to indicate the process by which the oscillations stop as velocity increases, although there is experimental evidence that this happens. Stability analysis of infinite free surfaces based upon inviscid flow provides a good estimate of the lower critical velocity[20]; however, an estimate of the upper critical velocity requires that the theory be based upon a continuous velocity profile and must include the influence of viscosity. Hence it seems reasonable to expect this added complication is needed to calculate the cessation of oscillation of cavities.

Further, the problem should be studied theoretically to determine the existence of other stable states. That is, it is known from the experiments that the unstable oscillations grow to a steady value at some finite amplitude. It would be of practical interest to be able to calculate this in detail.

Appendix A: Approximate Form for the Perturbation Term

At low Mach numbers the perturbation term can be written

$$\frac{\pi(l_x/l_z)q^2(-B + iA)}{2(A^2 - B^2)} \tag{A1}$$

where

$$A = (16S_1/\pi)^2 J_o(\omega l_x/a_\infty) - 4\pi S_1/(1 - M_\infty^2)^{1/2}(S_1^2 - \pi^2)$$

and

$$B = 8S_1/(1 - M_\infty^2)^{1/2}\,(S_1^2 - \pi^2) - (16/\pi)S_1^2 N_0(\omega l_x/a_\infty)$$

At large Mach numbers the theoretical value for the perturbation takes the form

$$[\pi l_x/l_z q^2 S_1^2(C + iD)]/2(D^2 - C^2) \tag{A2}$$

where

$$C = -\pi/S + (4/M_\infty)\sin S_1 - (8\cos S_1/S_1^2) \times (1 - \pi^2/S_1^2)(-\cos S_1 + \sin S_1/S_1^2) + 1$$

$$-D = 4 + (8/\pi)\sin S_1 + [(8\pi/S_1)\sin S_1] \times (1 - \pi^2/S_1^2) + (2/\pi S_1)\ln(S_1/M_\infty) - 0.232/\pi S_1$$

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